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Technical Notes

Heuristic Thermal Postbuckling and Large-Amplitude Vibration Formulations of Beams

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Nomenclature

A = area of cross section
b = maximum central deflection or amplitude
E = Young's modulus
I = area moment of inertia
L = length of the column or beam
P_L = mechanical equivalent of the thermal buckling load
P_{NL} = mechanical equivalent of the thermal postbuckling load
r = radius of gyration
T_a = tension developed in the beam due to large deflections
T_L = buckling temperature
T_{NL} = postbuckling temperature
u = axial displacement
w = transverse displacement
x = axial coordinate
α = coefficient of linear thermal expansion
ε_x = axial strain
λ_L = thermal buckling load parameter $(=P_{\mathrm{NL}}L^2/EI)$
λ_{NL} = thermal postbuckling load parameter, $P_L L^2/EI$
λ_{T_a} = axial tension parameter, $T_a L^2/EI$
ω_L = linear radian frequency
ω_{NL} = nonlinear radian frequency

Introduction

IMPORTANT nonlinear problems like the postbuckling behavior of the most widely used beams subjected to mechanical loads, where the nonlinearity exists in the moment-curvature relation, has been extensively studied in the classic books of Dym [1], Thompson and Hunt [2], and Timoshenko and Gere [3]. However, the postbuckling of beams with axially immovable ends subjected to thermal loads, as a result of which uniform compressive load is developed because of a uniform temperature rise from the stress-free temperature of the beam, did not receive much attention compared to the case of the mechanical loads. This problem is mainly governed by the nonlinearity involved in the stain-displacement relation, generally called the von-Kármán-type nonlinearity, applicable to beams, is presented for the first time by Rao and Raju [4] using the versatile finite element and the classical Reyleigh–Ritz methods. Recently, Shirong and Changjun [5] have considered the second-order nonlinearity, corresponding to the derivative of the axial displacement, in addition to the von-Kármán-type nonlinearity corresponding to the derivative of the lateral displacement while studying the thermal postbuckling behavior of beams more rigorously. It has been observed in the earlier studies that the postbuckling load-carrying capacity is higher in order of magnitude for the case of thermal loads compared to that of the mechanical loads, and the same trend is confirmed later for other structural members also. This phenomenon can be advantageously used particularly in the design of aerospace structures at high temperatures.

Similar studies on another equally important nonlinear problems like the large-amplitude vibration of beams with axially immovable ends are available where in the nonlinearity involved is again of the von Kármán type. The first classic study on this topic is due to Woinowsky-Krieger [6], where the tension, which is a measure of the nonlinearity, developed in the beam due to large deflections, is used to formulate the large-amplitude vibration problem of beams. Many researchers subsequently used the tension developed in the beam in their new formulations and a comprehensive bibliography on this topic is available in the book of Satyamoorthy [7]. Even in the first finite element formulation of Mei [8], the tension developed is used in the elements due to large deflections. However, the finite element formulation of Rao et al. [9] used a different approach by formulating this problem directly using the nonlinear strain-displacement relations and making some assumptions to simplify the problem. These assumptions and simplifications are eliminated in the finite element formulation of Singh et al. [10] by numerically integrating the temporal equation which resulted in obtaining very accurate results matching with those of Woinowsky-Krieger [6].

The aforementioned work on the thermal postbuckling and large-amplitude vibrations require a thorough knowledge of higher order mathematics. Though for the expert researchers in these fields this is not a problem, for the practicing engineers who aim at obtaining quick but accurate engineering solutions for these problems, the mathematical treatment involved is difficult. Consequently, they try to use the results available in the literature and very often these results may not be readily available to be applied for some specific practical structural members. As a consequence, some interpolation or extrapolation techniques are used, which sometimes give unreasonable results to these practical problems.

The aim of the present work is to develop independently very simple formulas for the thermal postbuckling and the large-amplitude free-vibration problems of beams. Some efforts in this direction can be seen in [11–13] for beams and other structural members. These simple formulations are based on the evaluation of the tension developed due to large deflections, following the procedure given in Woinowsky-Krieger [6]. However, there is a possibility for the less experienced in this field to think that the way the tension is calculated in [6], though straightforward, may give an impression that it is an average quantity, irrespective of the structural member considered. An attempt has been made to rigorously evaluate the tension developed by Rao and Reddy [14]. This attempt, though successful for the simply supported beam, did not work for the beams with other boundary conditions because of the incompleteness of the formulation. A successful attempt is made here to calculate the tension, with much less difficulty from the nonlinear strain-displacement relations of beam. Two standard uniform beam problems with either simply supported (SS) or clamped (CC) boundary conditions are considered in the present study. The numerical results obtained show the simplicity and advantage of the proposed heuristic formulation compared to the complex mathematical (rigorous) formulations. The limit of validity of this simple formulation is established by correlating the present results with

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those given in [5]. For a better appreciation of the utility of this note, the major contributions of the present work are as follows:

1) To derive the expression for the axial displacement and to evaluate its undetermined coefficient compatible with the lateral displacement function (one term admissible function) containing an undetermined coefficient.

2) To evaluate the axial tension developed using an alternate formulation, compared to the integral form for the same given by Woinowsky-Krieger [6] that can be used with equal ease, not only in the rectangular Cartesian coordinate system, but also in the polar coordinate system.

Evaluation of Tension

As the emphasis given in this note is to the beams, a detailed presentation of the theory for the evaluation of tension T_a developed due to large deflections is given here for the beam. However, the same procedure can also be judiciously extended for two-dimensional problems.

The nonlinear strain-displacement relation (von Kármán type) for the one-dimensional problem is

$$\epsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \tag{1}$$

Multiplying both sides of Eq. (1) by EA, we get

$$EA\,\epsilon_x = T_a = EA\left[\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right] \tag{2}$$

Since T_a is constant along the length of the beam, one can write $\mathrm{d}T_a/\mathrm{d}x = 0$ and, accordingly, Eq. (2) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right] = 0 \tag{3}$$

or

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) \tag{4}$$

Integrating Eq. (4) twice, after assuming an admissible function for w as

$$w = b\sin\frac{\pi x}{L} \tag{5}$$

which is an exact admissible function for the buckling and the first free-vibration modes of the SS beam and

$$w = \frac{b}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \tag{6}$$

which is an exact admissible function for the buckling mode and slightly approximate for the first free-vibration mode. It may be noted here that the admissible function is called exact, when substituted in the differential equation linear or nonlinear, it gets cancelled. These admissible functions are exact because of the form of nonlinear differential equations considered in this study. These are the commonly used trigonometric admissible functions [15], we get an admissible function for u, which is otherwise difficult to assume. Care has been taken in choosing the function w, so that the value of w is equal to b at the central point of the beam. Note that the constants of integration can be evaluated using the two boundary conditions (axially immovable conditions), given by

$$u(0) = u(L) = 0 \tag{7}$$

The admissible functions for u obtained using the present procedure for the simply supported and clamped conditions that are dependent of the beam configuration considered are

$$u = -\frac{b^2}{8}\sin\frac{2\pi x}{L} \tag{8}$$

for the SS beam and

$$u = \frac{b^2}{16}\sin\frac{4\pi x}{L} \tag{9}$$

for the CC beam.

Once the expression for u is known, the tension T_a developed in the beam due to large deflections can be obtained from Eq. (2) as

$$T_a = \frac{EAb^2\pi^2}{4L^2} \tag{10}$$

and in the nondimensional form

$$\lambda_T = \frac{1}{4}\pi^2\left(\frac{b}{r}\right)^2 \tag{11}$$

for both the boundary conditions considered. In the following sections, the thermal postbuckling behavior and large-amplitude vibration behavior are discussed independently.

Thermal Postbuckling

Once the tension developed in the beam due to large deflections is known, the thermal postbuckling behavior of the column in terms of $\lambda_{\mathrm{NL}}/\lambda_L$, following [11], is

$$\frac{\lambda_{\mathrm{NL}}}{\lambda_L} = \frac{P_{\mathrm{NL}}}{P_L} = \frac{T_{\mathrm{NL}}}{T_L} = 1 + \frac{\lambda_{Ta}}{\lambda_L} \tag{12}$$

for both the SS and CC beams, where λ_T, λ_L, and λ_{NL} are the nondimensional tension, thermal buckling, and postbuckling load parameters. The values of λ_L are π^2 and $4\pi^2$ for the simply supported and clamped beams, respectively [3]. The compressive load P developed, because of the axially immovable ends, due to the temperature raise of ΔT from the stress-free condition of the beam, is

$$P = EA\alpha\Delta T \tag{13}$$

Equation (13) is obtained by equating the longitudinal expansion due to the temperature rise and contraction due to applied compressive load P.

Large-Amplitude Vibration

A detailed formulation of the simplified approach to study the large-amplitude free vibrations for any mode of vibration is presented by Rao et al. [15]. Without going into details, the expressions for the ratios of nonlinear to linear frequencies can be written as [15]

$$\frac{\omega_{\mathrm{NL}}}{\omega_L} = \left(1 + \frac{3}{4}\frac{\lambda_{Ta}}{\lambda_L}\right)^{1/2} \tag{14}$$

The values of λ_L are the same for SS and CC beams as given earlier.

Numerical Results and Discussion

The values of $\lambda_{\mathrm{NL}}/\lambda_L (=T_{\mathrm{NL}}/T_L)$ and $\omega_{\mathrm{NL}}/\omega_L$ are calculated independently from the tension parameter λ_{Ta} derived directly from the nonlinear strain-displacement relations of the beam and the buckling load parameter λ_L for several maximum deflection/amplitude ratios b/r. Note that, in the present simple continuum analysis, the functional form for u and its undetermined coefficient are derived from the trigonometric admissible functions chosen for w. On the contrary, in the finite element formulation, the cubic polynomial distributions, taken for w and u with different generalized coordinates, for the simple beam element, implicitly take care of the compatibility of u with w.

Table 1 gives the ratios of the postbuckling and buckling temperatures (T_{NL}/T_L) for the simply supported and clamped beams obtained from the present analysis, along with those obtained from the finite element method, for several values of the central deflection parameters b/r. It is seen from this table that the results from the present analysis are exactly the same as those obtained using the

Table 1 Values (T_{NL}/T_L) for uniform columns

b/r	Simply supported		Clamped	
	Present work	Rao et al. [9]	Present work	Rao et al [9]
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.0100	1.0100	1.0025	1.0025
0.4	1.0400	1.0400	1.0100	1.0100
0.6	1.0900	1.0900	1.0225	1.0225
0.8	1.1600	1.1600	1.0400	1.0400
1.0	1.2500	1.2500	1.0625	1.0625
2.0	2.0000	2.0000	1.2500	1.2500
3.0	3.2500	3.2500	1.5625	1.5625

finite element analysis. This is due to the fact that the admissible functions assumed in the present study are exact [16]. Similar results for the ratios of the nonlinear to linear radian frequencies ω_{NL}/ω_L obtained from the present analysis are given in Table 2 along with those obtained from the rigorous finite element analysis. In this case of large-amplitude vibrations, the admissible function assumed for w, though exact [6], the final results of the value ω_{NL}/ω_L are in a small error because of the use of the harmonic balance method [17] used to obtain these ratios in closed form as a function of b/r. Whereas in the finite element method, the direct numerical integration method [10] is used in which the required accuracy of these ratios can be achieved in the digital form for each b/r. However, in the case of the clamped beam, the admissible function chosen for w, though approximate, is widely used by many researchers. The maximum error in the values of ω_{NL}/ω_L obtained from the present and the finite element analysis is around 0.8365% for simply supported beam and 0.8275% for the clamped for $b/r = 3.0$.

As has been already mentioned, the present work uses the very popular von-Kármán-type nonlinearity where the second-order nonlinear terms containing the square of the derivatives of the axial displacement with respect to the axial coordinate x are neglected. Shirong and Changjun [5] give an exact mathematical treatment where the second-order nonlinear term is included in the study of the thermal postbuckling of beams. Quantification of the thermal postbuckling results of beams on the effect of the second-order nonlinear term in von-Kármán-type nonlinear analysis is possible if the same results obtained by both geometric nonlinear formulations are available. This is achieved by correlation of the parameter m used in [5] and b/r used in the present work. The relation between m and b/r is worked out for the CC beam considered in [5] as $m = (2\pi^2/160)(b/r)$. Table 3 gives the postbuckling results obtained using both the geometric nonlinear formulations for several values of m. It can be seen from this table that both the theories give almost the same results for T_{NL}/T_L up to the value of $m = 1.0$ ($b/r = 8.1056$) with a difference of 1.05%. Beyond this value of m, the difference goes on increasing with increasing m with a difference of 6.12% for $m = 2.0$ ($b/r = 16.211$). This difference is not very high for all practical purposes. However, it can be confidently concluded that, up to $b/r = 8$, the difference between these two geometric nonlinear formulations is more or less insignificant and this b/r can be treated as the limit of validity for the CC beam with von-Kármán-type geometric nonlinear theory. Similar quantitative conclusions can also be made for the SS beam, but were not attempted, seeing the trend of the ratios of the axial end reaction and critical load given in [5]. However, one has to be careful in using the results obtained considering only the geometric nonlinearity, as, for higher b/r ratios, the material nonlinearity has to be considered and this may significantly change the trends of the thermal postbuckling and large-amplitude vibration results.

Table 2 Values of (ω_{NL}/ω_L) for uniform beams

b/r	Simply supported		Clamped	
	Present work	Singh et al. [10]	Present work	Singh et al. [10]
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.0037	1.0037	1.0008	1.0009
0.4	1.0148	1.0148	1.0037	1.0036
0.6	1.0331	1.0331	1.0083	1.0083
0.8	1.0583	1.0582	1.0148	1.0142
1.0	1.0897	1.0892	1.0231	1.0221
2.0	1.3228	1.3128	1.0897	1.0854
3.0	1.6393	1.6257	1.1923	1.1825

Table 3 Comparative study of T_{NL}/T_L for two geometric nonlinear theories for CC beam

m	b/r	T_{NL}/T_L		Percentage difference
		Present work	Shirong and Changjun [5]	
0.2	1.1624	1.1642	1.1638	−0.0343
0.4	3.2422	1.6570	1.6563	−0.0422
0.6	4.8634	2.4783	2.4816	0.1329
0.8	6.4855	3.6280	3.6464	0.5046
1.0	8.1056	5.1063	5.1604	1.0483
1.2	9.7268	6.9132	7.0365	1.7522
1.4	11.347	9.0485	9.2914	2.6142
1.6	12.969	11.512	11.946	3.6630
1.8	14.590	14.304	15.024	4.7923
2.0	16.211	17.425	18.560	6.1153

Conclusions

A simple method is developed in this Note to evaluate the tension induced in the beams, with axially immovable ends, because of large deflections from the nonlinear strain-displacement (von Kármán type) relation. This tension (or tension parameter) is used to independently evaluate the thermal postbuckling and large-amplitude vibration behavior of beams. A major advantage of the proposed method is that, if the admissible function for the lateral displacement is known, the admissible function for the axial displacement can be directly evaluated. This eliminates the difficulties involved in arriving at a suitable admissible function for the axial displacement, which depends on the admissible lateral displacement function. This advantage can be clearly seen in the rigorous finite element formulation; the distributions for the lateral axial displacements have to be separately chosen. Quantification of the validity regions of b/r for the thermal postbuckling problem solved considering the von-Kármán-type nonlinearity and rigorous geometric nonlinear formulations is made in the present work. The method presented is simple, and closed-form solutions that can be easily obtained with significantly less mathematical treatment compared with the existing rigorous continuum or numerical methods can be easily applied to beams with other boundary conditions with more complicating effects.

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